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1972 J. Phys. A: Gen. Phys. 5 348

(http://iopscience.iop.org/0022-3689/5/3/004)

J. Phys. A: Gen. Phys., Vol. 5, March 1972. Printed in Great Britain

Chiral currents as field variables in a semiclassical approach

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MS received 2 September 1971

Abstract. The ways in which the 4-vector currents and invariants carried by an 8-component spinor field can appear as dynamical variables are investigated in the framework of a c-number theory. Simple derivations are given to a number of identities containing space–time derivatives of the currents and invariants. Expressions for the energy–momentum tensor and other tensors, for lagrangian densities and the resulting field equations all show symmetries and simple transformation properties with respect to isospin and chiral $SU(2) \times SU(2)$ transformations which, through the tetrad structure of the chiral currents, are connected with space–time properties. Contributions to field equations from different parts of a Lagrangian are given in a form that remains valid in the presence of other interaction terms and exhibits the presence of tetrad fields. Different choices of independent variables lead to complementary descriptions.

1. Tetrad fields and invariants

In a previous paper (Valatin 1968, to be referred to as I)†, the relationships between the large number of 4-vector currents and invariants carried by an 8-component c-number spinor field were analysed, and some rather suggestive simple geometrical relationships were noticed. The 4-vector currents form a number of orthogonal tetrad frames, two of which especially have simple relationships with isospin properties. There are many symmetries, and the ratios of the invariants define geometrical angles and hyperbolic angles in space–time between the tetrads.

As a preliminary to an attempt to find and investigate new forms of a properly quantized theory, the present paper tries to combine the purely kinematical and local relationships discussed in I with more dynamical aspects related to a space–time variation of the fields, still within the framework of a c-number theory. The basic dynamical variables in terms of which the field equations are to be described will be the 4-vector currents and invariants. If one assumes simple Lagrangians for the spinor fields, one can explore the ways in which different aspects of such a description will fit together. To a large extent, it will be found arbitrary, how to select the dynamical variables from the large number of vector and scalar fields, and different choices will correspond to complementary descriptions. Simplicity arguments in describing interactions of a desired type will, however, indicate preferences in the choice of variables. They may also indicate reasons for the occurrence of certain types of interactions.

Free fields with zero rest mass show a large number of symmetries which can be broken by interactions, and special attention will be paid to the more symmetric expressions. The masses of particles could result in any case from spontaneous symmetry

† This paper should be consulted for a long list of references to previous work.

breaking in a quantized theory. The underlying group related to a tetrad field with a complex invariant is the group GL(2, C) of linear transformations in two dimensions with complex coefficients, and the corresponding group for two independent tetrad fields is GL(2, C) × GL(2, C). (There is also a wider group related to the six tetrads given in I.) This point of view will not be discussed in the present paper, but may be kept in mind in considering some of the expressions that follow. The notation, which in some places will distinguish the 3-dimensional rotations connected with isospin, could be made more symmetric with respect to the 4-dimensional internal transformations of the four tetrad vectors. The group SU(2) × SU(2) which is related to the 3-dimensional rotations appears as a subgroup of the larger group.

Some of the relationships will be straightforward generalizations of results of Aymard (1956, 1957) and Takabayasi (1958a, 1958b) who investigated the equations of a 4-component Dirac field in terms of a single tetrad. At the same time, they will show additional symmetries related to isospin. Many other people discussed previously equations for bilinear quantities which follow from the 4-component Dirac equation (see I for references).

Definitions and local relationships are spelt out in much more detail in I, only as much as is necessary should be recapitulated here. An 8-component spinor field can be described in the Weyl representation as

$$\Psi = (\psi_p, \psi_n) \tag{1a}$$

where the 4-component fields ψ_p, ψ_n are given by

$$\psi_p = (\xi_p, \tilde{\eta}_p) \qquad \psi_n = (\xi_n, \tilde{\eta}_n) \tag{1b}$$

in terms of 2-component spinors

$$\xi = (\xi'_1, \xi'_2) \qquad \tilde{\eta} = (-\eta'^*_2, \eta'^*_1) \qquad \eta = (\eta'_1, \eta'_2). \tag{1c}$$

The charge conjugate field Ψ_c can be obtained by an interchange $\xi_p \leftrightarrow \eta_p$, $\xi_n \leftrightarrow \eta_n$. A prime is used on the spin components in (1*c*), and an unprimed symbol ξ_κ will stand in the following for a 2-component spinor ξ_p or ξ_n. The label κ will indicate p or n and will denote isospin components.

In defining null vectors and invariants in terms of the field Ψ, some economy can be achieved by using the notation $\tau_{\kappa'\kappa}$ for the isospin matrices

$$\begin{aligned} \tau_{pp} &= \tau_p = \tfrac{1}{2}(1+\tau_3) & \tau_{nn} &= \tau_n = \tfrac{1}{2}(1-\tau_3) \\ \tau_{pn} &= \tau_+ = \tfrac{1}{2}(\tau_1+\mathrm{i}\tau_2) & \tau_{np} &= \tau_- = \tfrac{1}{2}(\tau_1-\mathrm{i}\tau_2). \end{aligned} \tag{1d}$$

With this notation, the sixteen null vectors discussed in I are given by

$$\begin{aligned} (R_{\kappa'\kappa})_\mu &= \bar{\Psi}\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\tau_{\kappa'\kappa}\Psi & (L_{\kappa\kappa'})_\mu &= \bar{\Psi}\gamma_\mu\tfrac{1}{2}(1-\gamma_5)\tau_{\kappa'\kappa}\Psi \\ (Z_{\kappa'\kappa})_\mu &= \bar{\Psi}\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\tau_{\kappa'\kappa}\Psi_c & (Z^*_{\kappa'\kappa})_\mu &= \bar{\Psi}_c\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\tau_{\kappa'\kappa}\Psi. \end{aligned} \tag{2a}$$

In terms of 2-component spinors and Pauli matrices σ_μ these have the simple form

$$\begin{aligned} (R_{\kappa'\kappa})_\mu &= \xi^*_\kappa\sigma_\mu\xi_{\kappa'} & (L_{\kappa'\kappa})_\mu &= \eta^*_\kappa\sigma_\mu\eta_{\kappa'} \\ (Z_{\kappa'\kappa})_\mu &= \xi^*_\kappa\sigma_\mu\eta_{\kappa'} & (Z^*_{\kappa'\kappa})_\mu &= \eta^*_\kappa\sigma_\mu\xi_{\kappa'}. \end{aligned} \tag{2b}$$

The complex conjugates of the null vectors are given by the relationships

$$(R_{\kappa'\kappa})^* = R_{\kappa\kappa'} \qquad (L_{\kappa'\kappa})^* = L_{\kappa\kappa'} \qquad (Z_{\kappa'\kappa})^* = Z^*_{\kappa\kappa'}. \tag{2c}$$

The scalar and pseudoscalar fields of interest are defined by the real and imaginary parts of the complex quantities

$$\tfrac{1}{2}\Omega_{\kappa'\kappa} = \bar{\Psi}\tfrac{1}{2}(1+\gamma_5)\tau_{\kappa'\kappa}\Psi \tag{3a}$$

$$\tfrac{1}{2}\Omega_R = \bar{\Psi}_c\tfrac{1}{2}(1+\gamma_5)\tau_+\Psi \qquad \tfrac{1}{2}\Omega_L = \bar{\Psi}\tfrac{1}{2}(1+\gamma_5)\tau_-\Psi_c. \tag{3b}$$

For $\Omega_{\kappa'\kappa}$, the notation

$$\Omega_{pp} = \Omega_p \qquad \Omega_{nn} = \Omega_n \qquad \Omega_{pn} = \Omega_+ \qquad \Omega_{np} = \Omega_- \tag{3c}$$

will also be used. In terms of 2-component spinors, one has

$$\tfrac{1}{2}\Omega_{\kappa'\kappa} = \eta_{\kappa'} \times \xi_\kappa \tag{3d}$$

$$\tfrac{1}{2}\Omega_R = \xi_p \times \xi_n \qquad \tfrac{1}{2}\Omega_L = \eta_n \times \eta_p \tag{3e}$$

where the exterior multiplication sign stands for the determinants of the spin components, and one has, for instance

$$\xi_p \times \xi_n = \begin{vmatrix} \xi'_{p1} & \xi'_{p2} \\ \xi'_{n1} & \xi'_{n2} \end{vmatrix}. \tag{3f}$$

With $\epsilon_{pn} = -\epsilon_{np} = 1, \epsilon_{pp} = \epsilon_{nn} = 0$, one can write (3$e$) in the form

$$\tfrac{1}{2}\epsilon_{\kappa'\kappa}\Omega_R = \xi_{\kappa'} \times \xi_\kappa \qquad \tfrac{1}{2}\epsilon_{\kappa'\kappa}\Omega_L = -\eta_{\kappa'} \times \eta_\kappa. \tag{3g}$$

A metric with $g_{00} = 1, g_{11} = g_{22} = g_{33} = -1$ is used for raising or lowering vector indices. These, however, will be omitted wherever possible, and V will stand for the vector V_μ and $V^2 = V^\mu V_\mu$ for its square. It has been shown in I that the sixteen null vectors define six tetrad fields, where each tetrad consists of four real vectors V, X, Y, W which have a length of the same absolute value and are orthogonal

$$V^2 = |\Omega|^2 \qquad X^2 = -|\Omega|^2 \qquad Y^2 = -|\Omega|^2 \qquad W^2 = -|\Omega|^2$$
$$VX = VY = VW = XY = YW = WX = 0. \tag{4a}$$

Each tetrad can be constructed from two 2-component spinors, and has accordingly a complementary tetrad from which it is independent, which is constructed from the other two spinors. In groups of such pairs, the six tetrads are given by the $6\times 4 = 24$ vectors

$$\begin{aligned} V_R &= R_{pp}+R_{nn} & V_L &= L_{pp}+L_{nn} \\ X_R &= R_{pn}+R_{np} & X_L &= L_{pn}+L_{np} \\ Y_R &= -\mathrm{i}(R_{pn}-R_{np}) & Y_L &= -\mathrm{i}(L_{pn}-L_{np}) \\ W_R &= R_{pp}-R_{nn} & W_L &= L_{pp}-L_{nn} \end{aligned} \tag{4b}$$

$$\begin{aligned} V_p &= R_{pp}+L_{pp} & V_n &= R_{nn}+L_{nn} \\ X_p &= Z_{pp}+Z^*_{pp} & X_n &= Z_{nn}+Z^*_{nn} \\ Y_p &= -\mathrm{i}(Z_{pp}-Z^*_{pp}) & Y_n &= -\mathrm{i}(Z_{nn}-Z^*_{nn}) \\ W_p &= R_{pp}-L_{pp} & W_n &= R_{nn}-L_{nn} \end{aligned} \tag{4c}$$

$$
\begin{aligned}
V_+ &= R_{nn}+L_{pp} & V_- &= R_{pp}+L_{nn}\\
X_+ &= Z_{np}+Z^*{}_{pn} & X_- &= Z_{pn}+Z^*{}_{np}\\
Y_+ &= -\mathrm{i}(Z_{np}-Z^*{}_{pn}) & Y_- &= -\mathrm{i}(Z_{pn}-Z^*{}_{np})\\
W_+ &= R_{nn}-L_{pp} & W_- &= R_{pp}-L_{nn}.
\end{aligned}
\tag{4d}
$$

Each set of tetrad vectors satisfies the relationships (4*a*) with the invariant Ω of the same subscript, according to the notation (3*c*, *d*, *e*).

The tetrad pairs (4*c*, *d*) depend through the null vectors Z on combinations of Ψ, $\bar{\Psi}$ with the charge conjugate fields $\bar{\Psi}_c$, Ψ_c, whereas the 4-vector fields of the tetrad pair (4*b*) are chiral currents with a more immediate interpretation. In terms of $\bar{\Psi}$, Ψ they are given by

$$
\begin{aligned}
(V_R)_\mu &= \bar{\Psi}\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\Psi & (V_L)_\mu &= \bar{\Psi}\gamma_\mu\tfrac{1}{2}(1-\gamma_5)\Psi\\
(X_R)_\mu &= \bar{\Psi}\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\tau_1\Psi & (X_L)_\mu &= \bar{\Psi}\gamma_\mu\tfrac{1}{2}(1-\gamma_5)\tau_1\Psi\\
(Y_R)_\mu &= \bar{\Psi}\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\tau_2\Psi & -(Y_L)_\mu &= \bar{\Psi}\gamma_\mu\tfrac{1}{2}(1-\gamma_5)\tau_2\Psi\\
(W_R)_\mu &= \bar{\Psi}\gamma_\mu\tfrac{1}{2}(1+\gamma_5)\tau_3\Psi & (W_L)_\mu &= \bar{\Psi}\gamma_\mu\tfrac{1}{2}(1-\gamma_5)\tau_3\Psi.
\end{aligned}
\tag{4e}
$$

Since the space-like vectors X_R, Y_R, W_R are orthogonal and have the same length and retain this property after an isospin transformation, the expressions (4*e*) relate the isospin transformations of isovectors with local space-like rotations of the three tetrad axes in space–time. Similar rotations apply for the tetrad vectors X_L, $-Y_L$, W_L. The two frames rotate independently with respect to chiral $\mathrm{SU(2)\times SU(2)}$ transformations. The definitions (2*b*), (4*b*) have been chosen in such a way that charge conjugation transforms X_R, Y_R, W_R into X_L, Y_L, W_L. The sign of Y_L in (4*e*) is the familiar minus sign which distinguishes G parity transformations from charge conjugation. Here it appears changing right-handedness into left-handedness in the local tetrad frames.

2. Energy–momentum tensor and Lagrangian

In analogy with the spinorial construction of the null vectors (2*b*), one can introduce the tensors

$$
\begin{aligned}
(\hat{\mathscr{T}}_{\kappa'\kappa}{}^{R})_{\mu\nu} &= \mathrm{i}\xi^*_{\kappa'}\sigma_\mu\nabla_\nu\xi_\kappa & (\hat{\mathscr{T}}_{\kappa'\kappa}{}^{L})_{\mu\nu} &= \mathrm{i}\eta^*_{\kappa'}\sigma_\mu\nabla_\nu\eta_\kappa\\
(\hat{\mathscr{T}}_{\kappa'\kappa}{}^{RL})_{\mu\nu} &= \mathrm{i}\xi^*_{\kappa'}\sigma_\mu\nabla_\nu\eta_\kappa & (\hat{\mathscr{T}}_{\kappa'\kappa}{}^{LR})_{\mu\nu} &= \mathrm{i}\eta^*_{\kappa'}\sigma_\mu\nabla_\nu\xi_\kappa.
\end{aligned}
\tag{5a}
$$

Both ∇_ν and ∂_ν will be used to denote partial differential operators.

The kinetic part of the energy–momentum tensor which contains derivatives of the spinor fields

$$
\mathscr{T}_{\mu\nu}{}^{(0)} = \frac{\hbar c}{2}\mathrm{i}(\bar{\Psi}\gamma_\mu\nabla_\nu\Psi-\nabla_\nu\bar{\Psi}\gamma_\mu\Psi)
\tag{5b}
$$

is given in terms of the tensors (5*a*) as

$$
\mathscr{T}_{\mu\nu}{}^{(0)} = \frac{\hbar c}{2}\{((\hat{\mathscr{T}}_{pp}{}^{R})_{\mu\nu}-(\hat{\mathscr{T}}_{pp}{}^{L})_{\mu\nu}+(\hat{\mathscr{T}}_{nn}{}^{R})_{\mu\nu}-(\hat{\mathscr{T}}_{nn}{}^{L})_{\mu\nu})+(\ldots)^*\}.
\tag{5c}
$$

Only this part of the energy–momentum tensor leads to any new type of expressions in terms of the tetrad vectors and invariants. The contribution of rest mass and interactions to the Lagrangian will be discussed later.

In order to express the tensor $\hat{\mathscr{T}}_{\kappa'\kappa}{}^{R}$ in terms of the invariant Ω_R and the null vectors of label R or the related tetrad vectors, one can proceed in two steps. First, one can exploit the fact that there are only two linearly independent 2-component spinors at a given point of space–time. If one chooses for these ξ_p and ξ_n, the two component spinors $\nabla_\nu \xi_\kappa$ for fixed ν can be expressed as linear combinations

$$\nabla\xi_\kappa = \beta^p{}_\kappa \xi_p + \beta^n{}_\kappa \xi_n. \tag{6a}$$

In this vectorial notation $\beta^p{}_\kappa$ stands for a vector with components $(\beta^p{}_\kappa)_\nu$. With upper and lower isospin indices defined by

$$\xi^p = \xi_n \qquad \xi^n = -\xi_p \tag{6b}$$

and with a summation convention, one can also write

$$\nabla\xi_\kappa = \beta^\lambda{}_\kappa \xi_\lambda. \tag{6c}$$

The coefficients $\beta^\lambda{}_\kappa$ can be determined by forming the exterior product of (6*a*) with ξ^λ. One term in the product vanishes, the other gives a contribution proportional to Ω_R according to the definition (3*d*), and one obtains

$$\beta^\lambda{}_\kappa = -\frac{2}{\Omega_R}\xi^\lambda \times \nabla\xi_\kappa. \tag{6d}$$

With the expression (6*c*) of $\nabla\xi_\kappa$ and the definition (5*a*) of $\hat{\mathscr{T}}_{\kappa'\kappa}{}^{R}$ one has

$$\hat{\mathscr{T}}_{\kappa'\kappa}{}^{R} = \mathrm{i}R_{\kappa'\kappa}\beta^\lambda{}_\kappa. \tag{6e}$$

As the second step in transforming $\hat{\mathscr{T}}_{\kappa'\kappa}{}^{R}$, one can use the identity (7*d*) of I which expresses the scalar product of two 4-vectors formed from arbitrary 2-component spinors in terms of the related invariants. According to this identity, one has

$$\begin{aligned}(\xi^*_{\lambda'}\sigma^\mu\xi_\lambda)(\nabla_\nu\xi^*_{\kappa'}\sigma_\mu\xi_\kappa) &= 2(\xi^*_{\lambda'}\times\nabla_\nu\xi^*_{\kappa'})(\xi_\lambda\times\xi_\kappa)\\ (\xi^*_{\lambda'}\sigma^\mu\xi_\lambda)(\xi^*_{\kappa'}\sigma_\mu\nabla_\nu\xi_\kappa) &= 2(\xi^*_{\lambda'}\times\xi^*_{\kappa'})(\xi_\lambda\times\nabla_\nu\xi_\kappa).\end{aligned} \tag{7a}$$

With the notation (2*b*), (3*f*) and $R^\mu R_\mu = RR$, the sum of the two relationships gives

$$R_{\lambda'\lambda}\nabla R_{\kappa'\kappa} = \epsilon_{\lambda\kappa}\Omega_R(\xi^*_{\lambda'}\times\nabla\xi^*_{\kappa'}) + \epsilon_{\lambda'\kappa'}\Omega^*_R(\xi_\lambda\times\nabla\xi_\kappa). \tag{7b}$$

In subtracting the identity obtained by an interchange of κ' and λ', one has

$$R_{\lambda'\lambda}\nabla R_{\kappa'\kappa} - R_{\kappa'\lambda}\nabla R_{\lambda'\kappa} = \epsilon_{\lambda\kappa}\Omega_R\nabla(\xi^*_{\lambda'}\times\xi^*_{\kappa'}) + \epsilon_{\lambda'\kappa'}2\Omega^*_R(\xi_\lambda\times\nabla\xi_\kappa). \tag{7c}$$

For $\lambda' = n$, $\kappa' = p$, $\epsilon_{np} = -1$, the left hand side can be written as $R^\sigma{}_\lambda\nabla R_{\sigma\kappa}$ and for the expression (6*d*) of $\beta^\lambda{}_\kappa$ one obtains

$$\beta^\lambda{}_\kappa = \frac{1}{|\Omega_R|^2}(-\delta^\lambda{}_\kappa\tfrac{1}{2}\Omega_R\nabla\Omega^*_R + R^{\sigma\lambda}\nabla R_{\sigma\kappa}). \tag{7d}$$

With the expression (6*e*) of $\hat{\mathscr{T}}_{\kappa'\kappa}{}^{R}$, this gives

$$\begin{aligned}\mathscr{T}^R_{\kappa'\kappa} + (\mathscr{T}^R_{\kappa\kappa'})^* = &-R_{\kappa'\kappa}\frac{1}{2\mathrm{i}}\frac{1}{|\Omega_R|^2}(\Omega^*_R\nabla\Omega_R - \Omega_R\nabla\Omega^*_R)\\ &+\mathrm{i}\frac{1}{|\Omega_R|^2}(R_{\kappa'\lambda}(R^{\sigma\lambda}\nabla R_{\sigma\kappa}) - R_{\sigma\kappa}(R^{\sigma\lambda}\nabla R_{\kappa'\lambda})).\end{aligned} \tag{8a}$$

In the last term, the first relationship (2*c*) has been used. From the definition (5*a*) of $\hat{\mathscr{T}}^R_{\kappa'\kappa}$ follows immediately the relationship:

$$-\mathrm{i}(\hat{\mathscr{T}}^R_{\kappa'\kappa}-(\hat{\mathscr{T}}^R_{\kappa\kappa'})^*)_{\mu\nu} = \partial_\nu(R_{\kappa'\kappa})_\mu. \tag{8b}$$

If one wants to verify this with the form (6*e*), (7*d*) of $\hat{\mathscr{T}}^R_{\kappa'\kappa}$, the identity

$$\frac{2}{|\Omega_R|^2}(R^{\kappa\lambda})^\mu(R_{\kappa\lambda})^\nu = g^{\mu\nu} \tag{8c}$$

is to be made use of. The content of this identity is more apparent in the form

$$\frac{1}{|\Omega_R|^2}(V^\mu V^\nu - X^\mu X^\nu - Y^\mu Y^\nu - W^\mu W^\nu)_R = g^{\mu\nu} \tag{8d}$$

which corresponds to a decomposition of the unit operator in terms of the projection operators of the four tetrad vectors.

The invariants

$$\hat{\mathscr{L}}^R_{\kappa'\kappa} = g^{\mu\nu}(\hat{\mathscr{T}}^R_{\kappa'\kappa})_{\mu\nu} = \mathrm{i}\xi^*_{\kappa'}\sigma^\mu\nabla_\mu\xi_\kappa \tag{9a}$$

and with similar definitions, the invariants

$$\hat{\mathscr{L}}^R_{\kappa'\kappa},\ \hat{\mathscr{L}}^L_{\kappa'\kappa},\ \hat{\mathscr{L}}^{RL}_{\kappa'\kappa},\ \hat{\mathscr{L}}^{LR}_{\kappa'\kappa} \tag{9b}$$

formed from the tensors (5*a*) differ from the related tensors only through an interchange $\sigma_\mu\nabla_\nu \leftrightarrow \sigma^\mu\nabla_\mu$. In the following, most of the expressions will be given for these invariant densities but, with a minor change in notation, they will obviously remain valid for the related tensors.

Together with the invariants

$$\mathscr{L}_{\kappa'\kappa} = \hat{\mathscr{L}}_{\kappa'\kappa}+(\hat{\mathscr{L}}_{\kappa\kappa'})^* \tag{9c}$$

it is convenient to define the related isoscalar and isovector quantities

$$\mathscr{L}_0 = \mathscr{L}_{pp}+\mathscr{L}_{nn}$$

$$\mathscr{L}_1 = \mathscr{L}_{pn}+\mathscr{L}_{np} \qquad \mathrm{i}\mathscr{L}_2 = \mathscr{L}_{pn}-\mathscr{L}_{np} \qquad \mathscr{L}_3 = \mathscr{L}_{pp}-\mathscr{L}_{nn}. \tag{9d}$$

With the notation

$$\Omega = |\Omega|\exp(\mathrm{i}\theta) \tag{10a}$$

$$\frac{1}{2\mathrm{i}}\frac{1}{|\Omega|^2}(\Omega^*\nabla\Omega-\Omega\nabla\Omega^*) = \nabla\theta \tag{10b}$$

the expressions (9*a*) (8*a*) give for these, in terms of the tetrad vectors

$$\mathscr{L}^R_0 = -\left(V\nabla\theta+\frac{1}{|\Omega|^2}(X(Y\nabla W)+W(X\nabla Y)+Y(W\nabla X))\right)^R \tag{11a}$$

$$\mathscr{L}^R_1 = -\left(X\nabla\theta+\frac{1}{|\Omega|^2}(Y(W\nabla V)+V(Y\nabla W)+W(V\nabla Y))\right)^R \tag{11b}$$

$$\mathscr{L}^R_2 = -\left(Y\nabla\theta+\frac{1}{|\Omega|^2}(W(X\nabla V)+V(W\nabla X)+X(V\nabla W))\right)^R \tag{11c}$$

$$\mathscr{L}^R_3 = -\left(W\nabla\theta+\frac{1}{|\Omega|^2}(X(Y\nabla V)+V(X\nabla Y)+Y(V\nabla X))\right)^R. \tag{11d}$$

Charge conjugation, that is an interchange $\xi \leftrightarrow \eta$, leads to $\hat{\mathscr{T}}^R_{\kappa'\kappa} \leftrightarrow \hat{\mathscr{T}}^L_{\kappa'\kappa}$, $\mathscr{L}^R_{\kappa'\kappa} \leftrightarrow \mathscr{L}^L_{\kappa'\kappa}$ and to

$$\mathscr{L}_0{}^R,\ \mathscr{L}_1{}^R,\ \mathscr{L}_2{}^R,\ \mathscr{L}_3{}^R \leftrightarrow \mathscr{L}_0{}^L,\ \mathscr{L}_1{}^L,\ \mathscr{L}_2{}^L,\ \mathscr{L}_3{}^L \tag{11e}$$

that is $\mathscr{L}_0{}^L$, $\mathscr{L}_1{}^L$, $\mathscr{L}_2{}^L$, $\mathscr{L}_3{}^L$ are given by the same expressions (11*a–d*) in terms of the tetrad vectors V_L, X_L, Y_L, W_L.

The derivative dependent kinetic part of the lagrangian density

$$\mathscr{L}^{(0)} = \frac{\hbar c}{2}\,\mathrm{i}(\bar{\Psi}\gamma^\mu\nabla_\mu\Psi - (\nabla_\mu\bar{\Psi})\gamma^\mu\Psi) \tag{12a}$$

can be written in terms of variables of the two tetrad fields of label R, L as

$$\mathscr{L}^{(0)} = \frac{\hbar c}{2}(\mathscr{L}_0{}^R - \mathscr{L}_0{}^L) \tag{12b}$$

with the expressions (11*a*) and (11*e*). The identity of the expressions (12*a*) and (12*b*) (11*a*) (11*e*) is independent of the dynamics, and is valid for the kinetic part of the Lagrangian, and the related energy–momentum tensor, in the presence of any additional interactions.

In the notation of the expressions (11*a–d*), vectors of the type $(X\nabla Y)$ stand for

$$(X\nabla Y)_\mu = X^\nu \partial_\mu Y_\nu. \tag{13a}$$

The trilinear expressions of tetrad vectors in $\mathscr{L}_j$ can be written in the alternative forms

$$X(Y\nabla W) + W(X\nabla Y) + Y(W\nabla X) = \frac{1}{2}\begin{vmatrix} X^\mu & Y^\mu & W^\mu \\ X^\nu & Y^\nu & W^\nu \\ \partial_\mu X_\nu & \partial_\mu Y_\nu & \partial_\mu W_\nu \end{vmatrix}$$

$$= \frac{1}{4}\begin{vmatrix} X^\mu & Y^\mu & W^\mu \\ X^\nu & Y^\nu & W^\nu \\ \partial_\mu X_\nu - \partial_\nu X_\mu & \partial_\mu Y_\nu - \partial_\nu Y_\mu & \partial_\mu W_\nu - \partial_\nu W_\mu \end{vmatrix}$$

$$= \tfrac{1}{4}\{(X^\mu Y^\nu - X^\nu Y^\mu)(\partial_\mu W_\nu - \partial_\nu W_\mu) + (Y^\mu W^\nu - Y^\nu W^\mu)(\partial_\mu X_\nu - \partial_\nu X_\mu)$$
$$+ (W^\mu X^\nu - W^\nu X^\mu)(\partial_\mu Y_\nu - \partial_\nu Y_\mu) \tag{13b}$$

which show their relationship with the Curl of the three tetrad vectors. With (11*a–d*), one can also write

$$\mathscr{L}_0 V - \mathscr{L}_1 X - \mathscr{L}_2 Y - \mathscr{L}_3 W = -|\Omega|^2\nabla\theta + \frac{1}{|\Omega|^2}Q \tag{13c}$$

where

$$Q^\alpha = \frac{1}{2}\begin{vmatrix} X^\alpha & Y^\alpha & W^\alpha & V^\alpha \\ X^\mu & Y^\mu & W^\mu & V^\mu \\ X^\nu & Y^\nu & W^\nu & V^\nu \\ \partial_\mu X_\nu & \partial_\mu Y_\nu & \partial_\mu W_\nu & \partial_\mu V_\nu \end{vmatrix}. \tag{13d}$$

With a label R, the 4-vector (13c) has components $|\Omega_R|\mathscr{L}_j^R$ along the orthogonal directions of the chiral currents V_R, X_R, Y_R, W_R. A chiral isospin transformation rotates the vectors X_R, Y_R, W_R locally in space–time, and an isospin transformation on $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ corresponds to taking the components of the 4-vector (13c) along the new axes.

3. Variational principle

A possible way for choosing the dynamical variables for a variational principle is to express the lagrangian density $\mathscr{L}$ in terms of the quantities of a pair of tetrad fields. The four real vectors of a tetrad have sixteen vector components which, together with the related complex invariant, correspond to eighteen real field quantities. Because of the orthonormality conditions (4a) of the tetrad vectors, however, only eight of these are independent. The supplementary conditions (4a) can be taken into account by adding to the lagrangian density multiplier terms of the form

$$\mathscr{L}^\lambda = \lambda_{VV}(|\Omega|^2 - V^2) + \lambda_{XX}(|\Omega|^2 + X^2) + \lambda_{YY}(|\Omega|^2 + Y^2) + \lambda_{WW}(|\Omega|^2 + W^2) \\ + \lambda_{XV} XV + \lambda_{YV} YV + \lambda_{WV} WV + \lambda_{XY} XY + \lambda_{YW} YW + \lambda_{WX} WX. \tag{14a}$$

If the supplementary conditions are satisfied $\mathscr{L}^\lambda$ vanishes, that is $\mathscr{L}^\lambda$ is weakly zero in the terminology of Dirac, which will be denoted by

$$\mathscr{L}^\lambda \simeq 0. \tag{14b}$$

If one takes as variables those of the tetrads with labels R and L, the variational principle reads

$$\delta \int \mathrm{d}^4 x (\mathscr{L} + \mathscr{L}^\lambda{}_R + \mathscr{L}^\lambda{}_L) = 0. \tag{14c}$$

Functional differential operators like $\delta/\delta V$ are usually applied on the action integral, in writing $\delta\mathscr{L}/\delta V$ they will stand for the variational derivatives $\partial/\partial V - \partial^\nu(\partial/\partial(\partial^\nu V))$.

As noted by Aymard (1956, 1957) in the case of a single tetrad of unit vectors, there are simple combinations of the variational derivatives which lead to a form of the lagrangian equations from which the multipliers λ have been eliminated. This is due to the simple form of the multiplier terms in the expression (14a) of $\mathscr{L}^\lambda$. Taking into account the supplementary conditions, one has for instance

$$X \frac{\delta}{\delta V} \mathscr{L}^\lambda \simeq \lambda_{XV} X^2 \tag{14d}$$

$$V \frac{\delta}{\delta X} \mathscr{L}^\lambda \simeq \lambda_{XV} V^2 \tag{14e}$$

and the sum of the two expressions is weakly zero. Through similar considerations, one has

$$\left(X\frac{\delta}{\delta V}+V\frac{\delta}{\delta X}\right)\mathscr{L}^{\lambda} \simeq 0 \tag{15a}$$

$$\left(Y\frac{\delta}{\delta V}+V\frac{\delta}{\delta Y}\right)\mathscr{L}^{\lambda} \simeq 0 \tag{15b}$$

$$\left(W\frac{\delta}{\delta V}+V\frac{\delta}{\delta W}\right)\mathscr{L}^{\lambda} \simeq 0 \tag{15c}$$

$$\left(Y\frac{\delta}{\delta W}-W\frac{\delta}{\delta Y}\right)\mathscr{L}^{\lambda} \simeq 0 \tag{15d}$$

$$\left(W\frac{\delta}{\delta X}-X\frac{\delta}{\delta W}\right)\mathscr{L}^{\lambda} \simeq 0 \tag{15e}$$

$$\left(X\frac{\delta}{\delta Y}-Y\frac{\delta}{\delta X}\right)\mathscr{L}^{\lambda} \simeq 0 \tag{15f}$$

and also

$$\left(V\frac{\delta}{\delta V}+X\frac{\delta}{\delta X}+Y\frac{\delta}{\delta Y}+W\frac{\delta}{\delta W}+\Omega\frac{\delta}{\delta\Omega}+\Omega^*\frac{\delta}{\delta\Omega^*}\right)\mathscr{L}^{\lambda} \simeq 0. \tag{15g}$$

The additional equation

$$\mathrm{i}\left(\Omega\frac{\delta}{\delta\Omega}-\Omega^*\frac{\delta}{\delta\Omega^*}\right)\mathscr{L}^{\lambda} = 0 \tag{15h}$$

holds as a strong identity, since $\mathscr{L}^{\lambda}$ depends only on $\Omega^*\Omega$, and does not depend on the phase of Ω. If one forms the variational derivatives of $\mathscr{L}$ in the combinations (15*a*–*h*), as will be done in the following, the multiplier terms can be altogether disregarded since the corresponding contributions vanish.

With the expression (12*b*) (11*a*) (11*e*) of the free Lagrangian, $\mathscr{L}^{(0)} \sim \mathscr{L}_0{}^R - \mathscr{L}_0{}^L$ has two parts each depending only on the variables of a single tetrad. The expressions (11*a*–*d*) show explicitly that the invariants $\mathscr{L}_1{}^R$, $\mathscr{L}_2{}^R$, $\mathscr{L}_3{}^R$ can be obtained from $\mathscr{L}_0{}^R$ by an interchange of the tetrad vector V with X or Y or W. One can see by simple inspection that the variational derivatives (15*a*–*c*), if applied on $\mathscr{L}_0{}^R$ result just in such an interchange, and one has accordingly

$$\left(X\frac{\delta}{\delta V}+V\frac{\delta}{\delta X}\right)^R\mathscr{L}_0{}^R = \mathscr{L}_0{}^R(V\leftrightarrow X) = \mathscr{L}_1{}^R \tag{16a}$$

$$\left(Y\frac{\delta}{\delta V}+V\frac{\delta}{\delta Y}\right)^R\mathscr{L}_0{}^R = \mathscr{L}_0{}^R(V\leftrightarrow Y) = \mathscr{L}_2{}^R \tag{16b}$$

$$\left(W\frac{\delta}{\delta V}+V\frac{\delta}{\delta W}\right)^R\mathscr{L}_0{}^R = \mathscr{L}_0{}^R(V\leftrightarrow W) = \mathscr{L}_3{}^R. \tag{16c}$$

The combination (15*g*) of variational derivatives gives the identity

$$\left(V\frac{\delta}{\delta V}+X\frac{\delta}{\delta X}+Y\frac{\delta}{\delta Y}+W\frac{\delta}{\delta W}+\Omega\frac{\delta}{\delta\Omega}+\Omega^*\frac{\delta}{\delta\Omega^*}\right)\mathscr{L}_0{}^R = \mathscr{L}_0{}^R. \tag{16d}$$

The four functional differential operators of the equations (15*d*, *e*, *f*, *h*), if applied to $\mathscr{L}_0{}^R$, give

$$\left(Y\frac{\delta}{\delta W}-W\frac{\delta}{\delta Y}\right)^R \mathscr{L}_0{}^R = \partial_\mu X^\mu{}_R \tag{17a}$$

$$\left(W\frac{\delta}{\delta X}-X\frac{\delta}{\delta W}\right)^R \mathscr{L}_0{}^R = \partial_\mu Y^\mu{}_R \tag{17b}$$

$$\left(X\frac{\delta}{\delta Y}-Y\frac{\delta}{\delta X}\right)^R \mathscr{L}_0{}^R = \partial_\mu W^\mu{}_R \tag{17c}$$

$$\mathrm{i}\left(\Omega\frac{\delta}{\delta \Omega}-\Omega^*\frac{\delta}{\delta \Omega^*}\right)^R \mathscr{L}_0{}^R = \partial_\mu V^\mu{}_R. \tag{17d}$$

By an interchange of labels R and L, the same expressions (16*a*–*d*), (17*a*–*d*) give the variational derivatives of $\mathscr{L}_0{}^L$ in terms of the related tetrad vectors and invariant.

The resulting expressions of the variational derivatives of

$$\mathscr{L}^{(0)} = \frac{\hbar c}{2}(\mathscr{L}_0{}^R - \mathscr{L}_0{}^L) \tag{18a}$$

remain valid in the presence of interaction terms. In the case of free massless fields, for which $\mathscr{L}^{(0)}$ is the total Lagrangian, the lagrangian field equations in terms of the two sets of tetrad variables R and L are

$$\mathscr{L}_0{}^R = 0 \qquad \mathscr{L}_1{}^R = 0 \qquad \mathscr{L}_2{}^R = 0 \qquad \mathscr{L}_3{}^R = 0 \tag{18b}$$

$$\partial_\mu V^\mu{}_R = 0 \qquad \partial_\mu X^\mu{}_R = 0 \qquad \partial_\mu Y^\mu{}_R = 0 \qquad \partial_\mu W^\mu{}_R = 0 \tag{18c}$$

and

$$\mathscr{L}_0{}^L = 0 \qquad \mathscr{L}_1{}^L = 0 \qquad \mathscr{L}_2{}^L = 0 \qquad \mathscr{L}_3{}^L = 0 \tag{18d}$$

$$\partial_\mu V^\mu{}_L = 0 \qquad \partial_\mu X^\mu{}_L = 0 \qquad \partial_\mu Y^\mu{}_L = 0 \qquad \partial_\mu W^\mu{}_L = 0. \tag{18e}$$

These are isoscalar and isovector equations, showing full symmetry also with respect to SU(2) × SU(2) transformations. With respect to the wider group of 2-dimensional unimodular transformations, one could consider them as four iso-4-vector equations. All the sixteen equations are scalars in space–time. On the other hand, with a construction similar to (13*c*, *d*), one could write the equations as four 4-vector equations in space–time which are invariant with respect to isospin transformations. The equations reveal a duality between space–time and isospin transformation properties.

If one expresses the free Lagrangian $\mathscr{L}^{(0)}$ in terms of the fields of the two independent tetrads with labels p and n, one has according to (5*b*, *c*), (9*a*), (12*a*)

$$\mathscr{L}^{(0)} = \frac{\hbar c}{2}(\mathscr{L}_3{}^p + \mathscr{L}_3{}^n) \tag{19a}$$

where $\mathscr{L}_3{}^p$ has the expression (11*d*) of $\mathscr{L}_3$ in terms of the tetrad quantities with label p. The previous combinations of variational derivatives in terms of these variables have to be applied on $\mathscr{L}_3$, and show less symmetry in the present notation. The massless free field equations have, however, the same form in terms of the tetrad quantities p, n

and are given by

$$\mathscr{L}_0{}^p = 0 \qquad \mathscr{L}_1{}^p = 0 \qquad \mathscr{L}_2{}^p = 0 \qquad \mathscr{L}_3{}^p = 0 \tag{19b}$$

$$\partial_\mu V^\mu{}_p = 0 \qquad \partial_\mu X^\mu{}_p = 0 \qquad \partial_\mu Y^\mu{}_p = 0 \qquad \partial_\mu W^\mu{}_p = 0 \tag{19c}$$

$$\mathscr{L}_0{}^n = 0 \qquad \mathscr{L}_1{}^n = 0 \qquad \mathscr{L}_2{}^n = 0 \qquad \mathscr{L}_3{}^n = 0 \tag{19d}$$

$$\partial_\mu V^\mu{}_n = 0 \qquad \partial_\mu X^\mu{}_n = 0 \qquad \partial_\mu Y^\mu{}_n = 0 \qquad \partial_\mu W^\mu{}_n = 0. \tag{19e}$$

In terms of the third independent tetrad pair with labels $+$, $-$, the free Lagrangian is

$$\mathscr{L}^{(0)} = \frac{\hbar c}{2}(\mathscr{L}_3{}^+ + \mathscr{L}_3{}^-) \tag{20a}$$

and the massless free field equations are

$$\mathscr{L}_0{}^+ = 0 \qquad \mathscr{L}_1{}^+ = 0 \qquad \mathscr{L}_2{}^+ = 0 \qquad \mathscr{L}_3{}^+ = 0 \tag{20b}$$

$$\partial_\mu V^\mu{}_+ = 0 \qquad \partial_\mu X^\mu{}_+ = 0 \qquad \partial_\mu Y^\mu{}_+ = 0 \qquad \partial_\mu W^\mu{}_+ = 0 \tag{20c}$$

$$\mathscr{L}_0{}^- = 0 \qquad \mathscr{L}_1{}^- = 0 \qquad \mathscr{L}_2{}^- = 0 \qquad \mathscr{L}_3{}^- = 0 \tag{20d}$$

$$\partial_\mu V^\mu{}_- = 0 \qquad \partial_\mu X^\mu{}_- = 0 \qquad \partial_\mu Y^\mu{}_- = 0 \qquad \partial_\mu W^\mu{}_- = 0. \tag{20e}$$

The three sets of equations in terms of the three tetrad pairs give complementary descriptions of the same field equations in terms of different basic variables.

For a 4-component spinor field and a single tetrad which would correspond here to the tetrad with label p, Aymard (1956, 1957) and Takabayasi (1958a, 1958b) have discussed essentially the same equations. The inclusion of isospin leads to a much larger number of equations with additional symmetries, and the discussed relationships between isospin and space–time transformation properties.

In the case of the 4-component spinor field, Takabayasi has pointed out that, for free massless fields, the tensors $\mathscr{T}_{\mu\nu}$ related to the invariants satisfy the conservation relationships

$$\partial^\mu \mathscr{T}_{\mu\nu} = 0. \tag{21a}$$

With isospin, the same relationships are valid for all the tensors

$$(\mathscr{T}_j{}^R)_{\mu\nu}, (\mathscr{T}_j{}^L)_{\mu\nu}, (\mathscr{T}_j{}^p)_{\mu\nu}, (\mathscr{T}_j{}^n)_{\mu\nu}, (\mathscr{T}_j{}^+)_{\mu\nu}, (\mathscr{T}_j{}^-)_{\mu\nu} \qquad j = 0, 1, 2, 3 \tag{21b}$$

related to the 24 invariants $\mathscr{L}_j$. The 4-vectors defined by the space integrals of these tensors are correspondingly constants of motion.

4. Vectorial and scalar interactions

A vector and pseudovector interaction can be introduced in the Lagrangian, conveniently expressed in terms of the tetrad vectors of label R and L as

$$\mathscr{L}_C = \frac{\hbar c}{2}(\mathscr{L}_C{}^R - \mathscr{L}_C{}^L) \tag{22a}$$

$$\mathscr{L}_C{}^R = C_0{}^R V_R + C_1{}^R X_R + C_2{}^R Y_R + C_3{}^R W_R \tag{22b}$$

$$\mathscr{L}_C{}^L = C_0{}^L V_L + C_1{}^L X_L + C_2{}^L Y_L + C_3{}^L W_L. \tag{22c}$$

Coupling constants are included in the definitions of the vector fields C_j, and a factor $\hbar c/2$ has been separated for easier comparison with previous expressions. Only given external fields C_j will be considered, but the resulting contributions to the field equations are very similar for current–current interactions.

The contribution of $\mathscr{L}_C{}^R$ to the Lagrangian equations is given by the expressions

$$\left(X\frac{\delta}{\delta V}+V\frac{\delta}{\delta X}\right)^R \mathscr{L}_C{}^R = C_0{}^R X_R + C_1{}^R V_R \tag{23a}$$

$$\left(Y\frac{\delta}{\delta V}+V\frac{\delta}{\delta Y}\right)^R \mathscr{L}_C{}^R = C_0{}^R Y_R + C_2{}^R V_R \tag{23b}$$

$$\left(W\frac{\delta}{\delta V}+V\frac{\delta}{\delta W}\right)^R \mathscr{L}_C{}^R = C_0{}^R W_R + C_3{}^R V_R \tag{23c}$$

$$\left(V\frac{\delta}{\delta V}+X\frac{\delta}{\delta X}+Y\frac{\delta}{\delta Y}+W\frac{\delta}{\delta W}+\Omega\frac{\delta}{\delta \Omega}+\Omega^*\frac{\delta}{\delta \Omega^*}\right)^R \mathscr{L}_C{}^R = \mathscr{L}_C{}^R \tag{23d}$$

$$\left(Y\frac{\delta}{\delta W}-W\frac{\delta}{\delta Y}\right)^R \mathscr{L}_C{}^R = C_3{}^R Y_R - C_2{}^R W_R \tag{24a}$$

$$\left(W\frac{\delta}{\delta X}-X\frac{\delta}{\delta W}\right)^R \mathscr{L}_C{}^R = C_1{}^R W_R - C_3{}^R X_R \tag{24b}$$

$$\left(X\frac{\delta}{\delta Y}-Y\frac{\delta}{\delta X}\right)^R \mathscr{L}_C{}^R = C_2{}^R X_R - C_1{}^R Y_R \tag{24c}$$

$$\mathrm{i}\left(\Omega\frac{\delta}{\delta \Omega}-\Omega^*\frac{\delta}{\delta \Omega^*}\right)\mathscr{L}_C{}^R = 0. \tag{24d}$$

The analogous variational derivatives of $\mathscr{L}_C{}^L$ are obtained immediately by an interchange of labels R and L. The eight Lorentz invariant field quantities (23*a*–*d*) (24*a*–*d*) given by these expressions again separate into two isoscalars and two isovectors, like the contributions from the free Lagrangians.

The vector and pseudovector interactions (22*b*, *c*) which involve the currents $\overline{\Psi}\gamma_\mu\tau_j\Psi$, $\overline{\Psi}\gamma_\mu\gamma_5\tau_j\Psi$ have a simple expression only in terms of the R, L tetrad vectors, and distinguish therefore these two tetrad fields as especially suitable to describe such interactions. Interactions of the same type would describe a gradient coupling with pseudoscalar or scalar fields. On the other hand, if one wants to introduce such scalar fields directly with scalar interactions, this is less easy to express in terms of the R, L field variables.

It is nonetheless instructive to investigate how such scalar interactions would present themselves in a variational principle in which the dynamical variables are the R, L fields. External scalar fields interacting with the scalar densities $\overline{\Psi}\tau_j\Psi$, $\overline{\Psi}\gamma_5\tau_j\Psi$ can be introduced by adding to the Lagrangian a term $(\hbar c/2)\mathscr{L}_\phi$, with

$$\mathscr{L}_\phi = \hat{\mathscr{L}}_\phi + (\hat{\mathscr{L}}_\phi)^* \tag{25a}$$

and

$$\hat{\mathscr{L}}_\phi = \phi^{*\kappa\kappa'}\Omega_{\kappa\kappa'}. \tag{25b}$$

The four complex invariants $\Omega_{\kappa\kappa'}$ have now to be expressed in terms of the fields of the

R, L tetrads. According to I(21) one has the identity

$$\Omega^*_{pn}\Omega^*_{np} - \Omega^*_{pp}\Omega^*_{nn} = \Omega^*_R\Omega^*_L \tag{26a}$$

between the six invariants Ω^*. This can be written alternatively as

$$\Omega^{*\lambda\lambda'}\Omega^*_{\lambda\lambda'} = -2\Omega^*_R\Omega^*_L. \tag{26b}$$

If one multiplies this identity with the square of $\Omega_{\kappa\kappa'}$, where no summation is implied with respect to $\kappa\kappa'$, one obtains

$$(\Omega_{\kappa\kappa'})^2 = (-2\Omega^*_R\Omega^*_L)^{-1}(\Omega^{*\lambda\lambda'}\Omega_{\kappa\kappa'})(\Omega^*_{\lambda\lambda'}\Omega_{\kappa\kappa'}). \tag{26c}$$

If one substitutes for the last two factors

$$\Omega^*_{\lambda\lambda'}\Omega_{\kappa\kappa'} = 2(L_{\lambda\kappa}R_{\lambda'\kappa'}) \tag{26d}$$

which is a special case of the identity I(7*d*), the square root of the expression (26*c*) gives $\Omega_{\kappa\kappa'}$ as a function of Ω^*_R, Ω^*_L and of the null vectors R^μ, L^μ, and consequently in terms of the R, L tetrad variables.

With the expression (26*c*), the variational derivative of $\Omega_{\kappa\kappa'}$ with respect to any of the R, L tetrad vectors, say A, can be written as

$$\frac{\delta\Omega_{\kappa\kappa'}}{\delta A} = \frac{1}{2\Omega_{\kappa\kappa'}}\frac{\delta(\Omega_{\kappa\kappa'})^2}{\delta A} = (-2\Omega^*_R\Omega^*_L)^{-1}\Omega^{*\lambda\lambda'}\frac{\delta(\Omega^*_{\lambda\lambda'}\Omega_{\kappa\kappa'})}{\delta A}. \tag{26e}$$

This, combined with the identity (26*d*), can be used to obtain the relevant combinations of variational derivatives of $\mathscr{L}_\phi$. The results are simplest if expressed again in terms of the complex invariants. They are given below in terms of the invariants

$$\begin{aligned}
\Omega_0 &= \Omega_{pp}+\Omega_{nn} = \bar{\Psi}(1+\gamma_5)\Psi \\
\Omega_1 &= \Omega_{pn}+\Omega_{np} = \bar{\Psi}(1+\gamma_5)\tau_1\Psi \\
\Omega_2 &= -\mathrm{i}(\Omega_{pn}-\Omega_{np}) = \bar{\Psi}(1+\gamma_5)\tau_2\Psi \\
\Omega_3 &= \Omega_{pp}-\Omega_{nn} = \bar{\Psi}(1+\gamma_5)\tau_3\Psi
\end{aligned} \tag{27}$$

and of similar combinations of the ϕ^* fields. In this way the isoscalar and isovector character of the resulting expressions is more apparent. One finds

$$\left(X\frac{\delta}{\delta V}+V\frac{\delta}{\delta X}\right)^R \mathscr{L}_\phi = \mathrm{Re}(\phi^*_0\Omega_1+\phi^*_1\Omega_0)+\mathrm{Im}(\phi^*_2\Omega_3-\phi^*_3\Omega_2) \tag{28a}$$

$$\left(Y\frac{\delta}{\delta V}+V\frac{\delta}{\delta Y}\right)^R \mathscr{L}_\phi = \mathrm{Re}(\phi^*_0\Omega_2+\phi^*_2\Omega_0)+\mathrm{Im}(\phi^*_3\Omega_1-\phi^*_1\Omega_3) \tag{28b}$$

$$\left(W\frac{\delta}{\delta V}+V\frac{\delta}{\delta W}\right)^R \mathscr{L}_\phi = \mathrm{Re}(\phi^*_0\Omega_3+\phi^*_3\Omega_0)+\mathrm{Im}(\phi^*_1\Omega_2-\phi^*_2\Omega_1) \tag{28c}$$

$$\begin{aligned}
&\left(V\frac{\delta}{\delta V}+X\frac{\delta}{\delta X}+Y\frac{\delta}{\delta Y}+W\frac{\delta}{\delta W}+\Omega\frac{\delta}{\delta\Omega}+\Omega^*\frac{\delta}{\delta\Omega^*}\right)^R\mathscr{L}_\phi \\
&\quad = \mathrm{Re}(\phi^*_0\Omega_0+\phi^*_1\Omega_1+\phi^*_2\Omega_2+\phi^*_3\Omega_3)
\end{aligned} \tag{28d}$$

$$\left(Y\frac{\delta}{\delta W}-W\frac{\delta}{\delta Y}\right)^R \mathscr{L}_\phi = \mathrm{Im}(\phi_0^*\Omega_1+\phi_1^*\Omega_0)-\mathrm{Re}(\phi_2^*\Omega_3-\phi_3^*\Omega_2) \tag{29a}$$

$$\left(W\frac{\delta}{\delta X}-X\frac{\delta}{\delta W}\right)^R \mathscr{L}_\phi = \mathrm{Im}(\phi_0^*\Omega_2+\phi_2^*\Omega_0)-\mathrm{Re}(\phi_3^*\Omega_1-\phi_1^*\Omega_3) \tag{29b}$$

$$\left(X\frac{\delta}{\delta Y}-Y\frac{\delta}{\delta X}\right)^R \mathscr{L}_\phi = \mathrm{Im}(\phi_0^*\Omega_3+\phi_3^*\Omega_0)-\mathrm{Re}(\phi_1^*\Omega_2-\phi_2^*\Omega_1) \tag{29c}$$

$$\mathrm{i}\left(\Omega\frac{\delta}{\delta \Omega}-\Omega^*\frac{\delta}{\delta \Omega^*}\right)^R \mathscr{L}_\phi = \mathrm{Im}(\phi_0^*\Omega_0+\phi_1^*\Omega_1+\phi_2^*\Omega_2+\phi_3^*\Omega_3) \tag{29d}$$

where Re and Im stand for the real and imaginary part of a complex quantity. Similar expressions are obtained for the variational derivatives of $\mathscr{L}_\phi$ with respect to the L tetrad variables.

The 16 real equations which describe vectorial and scalar interactions, as obtained from the variational principle in terms of the R, L tetrad variables, are given by equating with zero the sums of the three contributions (16*a–d*) (17*a–d*), (23*a–d*) (24*a–d*), (28*a–d*), (29*a–d*) and the sums of similar contributions of L tetrad derivatives. Similar equations can be obtained if one starts from a variational principle in terms of any of the other two tetrad pairs. A greater formal symmetry of the equations in terms of the six tetrad fields would result if one introduced additional vector fields coupled to the tetrad vectors formed from the null vectors Z and scalar fields coupled to Ω_R, Ω_L. All these equations could be obtained more directly from the Dirac equations of the spinor fields. The only nontrivial transformations in the course of such a derivation would be those of the kinetic contributions $\mathscr{L}_j^R$, $\mathscr{L}_j^L$, $\mathscr{L}_j^p$, $\mathscr{L}_j^n$, $\mathscr{L}_j^+$, $\mathscr{L}_j^-$ which have been discussed in detail.

The effect of a common rest mass m on the free field equations (18*b–e*) can be seen by choosing the external fields zero except for a constant real scalar field given by

$$\phi_0 = \phi_0^* = -\mu = -\frac{mc}{\hbar} \tag{30a}$$

which corresponds to

$$\frac{\hbar c}{2}\mathscr{L}_\phi = -mc^2\overline{\Psi}\Psi. \tag{30b}$$

This gives contributions

$$\mathrm{Re}\,\phi_0^*\Omega_j = -\mu\tfrac{1}{2}(\Omega_j+\Omega_j^*) \tag{30c}$$

$$\mathrm{Im}\,\phi_0^*\Omega_j = -\mu\frac{1}{2\mathrm{i}}(\Omega_j-\Omega_j^*) \tag{30d}$$

to the equations (18*b*, *c*), whereas equations (18*d*, *e*) obtain the same contributions with a minus sign for $j = 0, 1, 3$ and with a plus sign for $j = 2$. The definition of the R, L tetrad vectors and of $\mathscr{L}_j^R$, $\mathscr{L}_j^L$ was chosen in such a way that they are interchanged by an interchange $\xi \leftrightarrow \eta$ corresponding to charge conjugation, whereas one has at the same time $\Omega_j \leftrightarrow -\Omega_j$ for $j = 0, 1, 3$ and $\Omega_2 \leftrightarrow \Omega_2$. The origin of different signs for $j = 2$ has been commented upon in connection with the expressions (4*e*) of the R, L tetrad vectors.

The sum and difference of the resulting equations can be expressed in terms of the fields

$$(\mathscr{T}_j)^{\mu}{}_{\mu} = \mathscr{L}_j^R - \mathscr{L}_j^L \qquad j = 0, 1, 3 \qquad (\mathscr{T}_2)^{\mu}{}_{\mu} = \mathscr{L}_2{}^R + \mathscr{L}_2{}^L \tag{31a}$$

$$(\bar{\mathscr{T}}_j)^{\mu}{}_{\mu} = \mathscr{L}_j^R + \mathscr{L}_j^L \qquad j = 0, 1, 3 \qquad (\bar{\mathscr{T}}_2)^{\mu}{}_{\mu} = \mathscr{L}_2{}^R - \mathscr{L}_2{}^L \tag{31b}$$

$$\mathscr{V}_0 = V_R + V_L \qquad \mathscr{V}_1 = X_R + X_L \qquad \mathscr{V}_3 = W_R + W_L \qquad \mathscr{V}_2 = Y_R - Y_L \tag{31c}$$

$$\mathscr{A}_0 = V_R - V_L \qquad \mathscr{A}_1 = X_R - X_L \qquad \mathscr{A}_3 = W_R - W_L \qquad \mathscr{A}_2 = Y_R + Y_L \tag{31d}$$

which correspond to the $\bar{\Psi}$, Ψ expressions

$$(\mathscr{T}_j)_{\mu\nu} = \mathrm{i}(\bar{\Psi}\gamma_\mu\tau_j\partial_\nu\Psi - (\partial_\nu\bar{\Psi})\gamma_\mu\tau_j\Psi) \tag{32a}$$

$$(\bar{\mathscr{T}}_j)_{\mu\nu} = \mathrm{i}(\bar{\Psi}\gamma_\mu\gamma_5\tau_j\partial_\nu\Psi - (\partial_\nu\bar{\Psi})\gamma_\mu\gamma_5\tau_j\Psi) \tag{32b}$$

$$(\mathscr{V}_j)_\mu = \bar{\Psi}\gamma_\mu\tau_j\Psi \qquad (\mathscr{A}_j)_\mu = \bar{\Psi}\gamma_\mu\gamma_5\tau_j\Psi \tag{32c, d}$$

$$\omega_j = \bar{\Psi}\tau_j\Psi \qquad \bar{\omega}_j = -\mathrm{i}\bar{\Psi}\gamma_5\tau_j\Psi. \tag{32e}$$

One obtains

$$(\mathscr{T}_j)^{\mu}{}_{\mu} = 2\mu\omega_j \qquad (\bar{\mathscr{T}}_j)^{\mu}{}_{\mu} = 0 \tag{33a}$$

$$\partial^{\mu}(\mathscr{V}_j)_\mu = 0 \qquad \partial^{\mu}(\mathscr{A}_j)_\mu = 2\mu\bar{\omega}_j \tag{33b}$$

which are 16 real equations fully equivalent to the free 8-component Dirac equation. The nonconservation of axial charges which appears in the equations (33*b*) through the symmetry breaking connected with a rest mass is rather suggestive, as possibly connected with the origins of PCAC theorems and encouraging an attempt to relate the pseudoscalar fields $\bar{\omega}_j$ with interpolating meson fields.

5. An alternative choice of basic variables

The tetrad structure of the vectors and invariants carried by an 8-component spinor field Ψ makes it a natural choice to take the fields of two independent tetrads as dynamical variables. With the choice of the R, L tetrads, it was found that direct scalar interactions or rest mass terms result in contributions to the field equations which are simple only in terms of the invariants Ω. According to the expressions (26*c*, *d*), one can express these invariants in terms of the tetrad quantities, but only at the expense of introducing square roots. Alternatively, one could keep the invariants Ω as additional field variables, related to the R, L tetrad variables by identities, and add equations describing the space–time variation of the Ω fields. Since the number of vector and scalar fields carried by the spinor field Ψ is much larger than the number of independent variables, there are many ways to express the equations in terms of a minimal or redundant set of fields.

Another choice to be sketched briefly is to take the fields of one tetrad which represent eight independent real data, together with the eight real fields given by four complex invariants. If one selects the R tetrad which is given by the two spinors ξ_p, ξ_n, all other 4-vectors can be expressed as linear combinations of the R tetrad vectors with coefficients given by the invariants, in replacing the spinors η_p, η_n in their definitions by linear combinations

$$\eta_p = \alpha_p{}^p\xi_p + \alpha_p{}^n\xi_n \tag{34a}$$

$$\eta_n = \alpha_n{}^p\xi_p + \alpha_n{}^n\xi_n. \tag{34b}$$

This has been discussed in some detail in I. In writing (34*a*, *b*) as

$$\eta_\kappa = \alpha_\kappa{}^{\kappa'}\xi_{\kappa'} \tag{34c}$$

an exterior multiplication with ξ^κ gives

$$\alpha_\kappa{}^{\kappa'} = \frac{\Omega_\kappa{}^{\kappa'}}{\Omega_R} \tag{34d}$$

with

$$\tfrac{1}{2}\Omega_\kappa{}^{\kappa'} = \eta_\kappa \times \xi^{\kappa'}. \tag{34e}$$

All interaction terms without derivatives can then be expressed in terms of the fields of the R tetrad and the four invariants $\Omega_\kappa{}^{\kappa'}$.

In order to transform the field equations to this set of variables, the only new expressions to be found are those of $\mathscr{L}_j{}^L$. With

$$\hat{\mathscr{L}}^R{}_{\kappa\kappa'} = \mathrm{i}\xi^*_{\kappa'}\sigma^\mu\partial_\mu\xi_\kappa$$

the substitution (34*c*) gives

$$\hat{\mathscr{L}}_{\kappa\kappa'}{}^L = \mathrm{i}\eta^*_{\kappa'}\sigma^\mu\partial_\mu\eta_\kappa = \mathscr{A}_{\kappa'\kappa}{}^{\bar\kappa'\bar\kappa}\hat{\mathscr{L}}_{\bar\kappa\bar\kappa'}{}^R + (\hat{\mathscr{B}}_{\kappa'\kappa}{}^{\bar\kappa\bar\kappa'})^\mu(R_{\bar\kappa'\bar\kappa})_\mu \tag{35a}$$

where

$$\mathscr{A}_{\kappa'\kappa}{}^{\bar\kappa'\bar\kappa} = (\mathscr{A}_{\kappa\kappa'}{}^{\bar\kappa\bar\kappa'})^* = \alpha^{*\bar\kappa'}_{\kappa'}\alpha_\kappa{}^{\bar\kappa} \tag{35b}$$

$$(\hat{\mathscr{B}}_{\kappa'\kappa}{}^{\bar\kappa'\bar\kappa})^\mu = \mathrm{i}\alpha^{*\bar\kappa'}_{\kappa'}\partial^\mu\alpha_\kappa{}^{\bar\kappa}. \tag{35c}$$

This, together with the expression of $\hat{\mathscr{L}}^R_{\bar\kappa\bar\kappa'}$, in terms of the R tetrad variables, gives $\hat{\mathscr{L}}^L_{\kappa\kappa'}$ in function of the latter and the four invariants $\Omega_\kappa{}^{\kappa'}$. Note that one has

$$-\mathrm{i}(\hat{\mathscr{B}}_{\kappa'\kappa}{}^{\bar\kappa'\bar\kappa} - (\hat{\mathscr{B}}_{\kappa\kappa'}{}^{\bar\kappa\bar\kappa'})^*)^\mu = \partial^\mu\mathscr{A}_{\kappa'\kappa}{}^{\bar\kappa'\bar\kappa} \tag{35d}$$

whereas

$$\begin{aligned}(\mathscr{B}_{\kappa'\kappa}{}^{\bar\kappa'\bar\kappa})^\mu &= (\hat{\mathscr{B}}_{\kappa'\kappa}{}^{\bar\kappa'\bar\kappa} + (\hat{\mathscr{B}}_{\kappa\kappa'}{}^{\bar\kappa\bar\kappa'})^*)^\mu \\ &= \mathrm{i}(\alpha^{*\bar\kappa'}_{\kappa'}\partial^\mu\alpha_\kappa{}^{\bar\kappa} - \partial^\mu\alpha^{*\bar\kappa'}_{\kappa'}\alpha_\kappa{}^{\bar\kappa}).\end{aligned} \tag{35e}$$

From the expression of $\hat{\mathscr{L}}^L_{\kappa\kappa'}$ one obtains

$$\mathscr{L}^L_{\kappa'\kappa} = \hat{\mathscr{L}}^L_{\kappa'\kappa} + (\hat{\mathscr{L}}^L_{\kappa\kappa'})^* = \mathscr{A}_{\kappa'\kappa}{}^{\bar\kappa'\bar\kappa}\mathscr{L}^R_{\bar\kappa'\bar\kappa} + (\mathscr{B}_{\kappa'\kappa}{}^{\bar\kappa'\bar\kappa})^\mu(R_{\bar\kappa'\bar\kappa})_\mu \tag{36a}$$

and

$$\partial^\mu(L_{\kappa'\kappa})_\mu = -\mathrm{i}(\hat{\mathscr{L}}^L_{\kappa'\kappa} - (\hat{\mathscr{L}}^L_{\kappa\kappa'})^*) = \partial^\mu(\mathscr{A}_{\kappa'\kappa}{}^{\bar\kappa'\bar\kappa}R_{\bar\kappa'\bar\kappa})_\mu \tag{36b}$$

which, with the notation (9*d*) and a suitable 4-dimensional summation convention, can be written in the more transparent form

$$\mathscr{L}_j{}^L = \mathscr{A}_j{}^{\bar j}\mathscr{L}_{\bar j}{}^R + (\mathscr{B}_j{}^{\bar j})^\mu(R_{\bar j})_\mu \tag{36c}$$

$$\partial^\mu(L_j)_\mu = \partial^\mu(\mathscr{A}_j{}^{\bar j}R_{\bar j})_\mu. \tag{36d}$$

The last relationship is an immediate consequence of

$$L_j = \alpha_j^{\,j} R_j. \tag{36e}$$

The vectors L_j, R_j, $j = 0, 1, 2, 3$ denote in the last equations the tetrad vectors V, X, Y, W with label L or R.

With the angles θ and hyperbolic angles χ defined through

$$\Omega = \exp(\chi)\exp(i\theta)$$

the coefficients α given by the ratios (34*d*) can be expressed in terms of the differences of angles and hyperbolic angles which, according to I, represent actual geometrical angles between the tetrad fields. The 4-vectors $\mathscr{B}^\mu$ contain these angles and their gradients.

Whether the simple geometrical facts related to these angles analysed in I correspond to deeper physical relationships is not brought to the surface by the considerations of the present paper. Note, however, the suggestive fact (see the expressions (11*a*, *d*)) that the decompositions (12*a*, *b*), (19*a*), (20*a*) of the free Lagrangian contain the fields θ in the form of gradient coupling terms

$$\begin{array}{ll} (V_R)^\mu \partial_\mu \theta_R & (V_L)^\mu \partial_\mu \theta_L \\ (W_p)^\mu \partial_\mu \theta_p & (W_n)^\mu \partial_\mu \theta_n \\ (W_+)^\mu \partial_\mu \theta_+ & (W_-)^\mu \partial_\mu \theta_- . \end{array} \tag{37}$$

A vector interaction term $(V_R)^\mu (C_0{}^R)_\mu$ is of the same form as $(V_R)^\mu \partial_\mu \theta_R$, and the combination $(C_0{}^R)_\mu + \partial_\mu \theta_R$ appears in the sum of the two.

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